United Kingdom
Mathematics Trust

Junior Mathematical Olympiad

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Solutions

Section A

A1. What is the integer nearest to $\dfrac{59}{13}$?

SOLUTION $\boxed{\mathbf{5}}$

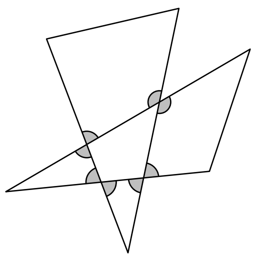
As $59 \div 13 = 4$ remainder 7, $\dfrac{59}{13} = 4\dfrac{7}{13}$ which is greater than $4\dfrac{1}{2}$ and smaller than 5.
So the integer nearest to $\dfrac{59}{13}$ is 5.

A2. What is the solution of the equation $24 \div (3 \div 2) = (24 \div 3) \div m$?

SOLUTION $\boxed{\dfrac{1}{2}}$

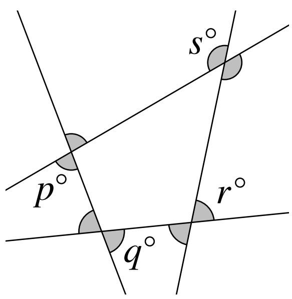
The left-hand side of the equation is $24 \div (3 \div 2) = 24 \times (2 \div 3) = 48 \div 3 = 16$.
The right-hand side of the equation is $(24 \div 3) \div m = 8 \div m$.
Hence $8 \div m = 16$. So $16m = 8$ and therefore $m = \dfrac{1}{2}$.

A3. Two triangles are drawn so that they overlap as shown.
What is the sum of the marked angles?



SOLUTION $\boxed{\mathbf{720^\circ}}$

Note that the angles marked $p^\circ, q^\circ, r^\circ, s^\circ$ are the four exterior angles of the quadrilateral in the centre of the diagram.
Therefore $p + q + r + s = 360$. Note also that the marked angles in the diagram form four pairs of vertically opposite angles with each pair containing one of the exterior angles of the quadrilateral.
As vertically opposite angles are equal, the sum, in degrees, of the marked angles is $2(p + q + r + s) = 2 \times 360 = 720$.



A4. What is the value of $\dfrac{(1^2+1)(2^2+1)(3^2+1)}{(2^2-1)(3^2-1)(4^2-1)}$? Give your answer in its simplest form.

SOLUTION $\boxed{\dfrac{\mathbf{5}}{\mathbf{18}}}$

The value of $\dfrac{(1^2+1)(2^2+1)(3^2+1)}{(2^2-1)(3^2-1)(4^2-1)}$ is $\dfrac{2\times5\times10}{3\times8\times15} = \dfrac{5\times4\times5}{3\times2\times4\times3\times5} = \dfrac{5}{3\times2\times3}$.

So the required value is $\dfrac{5}{18}$.

A5. A number line starts at -55 and ends at 55. If we start at -55, what percentage of the way along is the number 5.5?

SOLUTION **55%**

The distance from -55 to 55 is 110. The distance from -55 to 0 is 55 and from 0 to 5.5 it is 5.5. Therefore the distance from -55 to 5.5 is $55+5.5$.

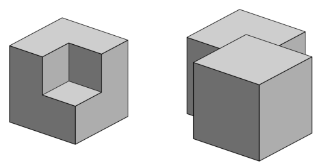
As a percentage of the total distance this is $\dfrac{55+5.5}{110}\times100 = \dfrac{55+5.5}{11}\times10 = (5+0.5)\times10$ $= 50+5 = 55$.

A6. Tea and a cake cost £4.50. Tea and an éclair cost £4. A cake and an éclair cost £6.50. What is the cost of tea, a cake and an éclair?

SOLUTION **£7.50**

Let the costs, in pounds, of tea, a cake and an éclair be t, c and e respectively.
Then $t+c=4.5$, $t+e=4$ and $c+e=6.5$. Adding these three equations gives $2t+2c+2e=15$.
So the cost of tea, a cake and an éclair is £15 ÷ 2 = £7.50.

A7. A $2\times2\times2$ cm cube has a $1\times1\times1$ cm cube removed from it to form the shape shown in the left-hand diagram. One of these shapes is inverted and put together with a second of the shapes on a flat surface, as shown in the right-hand diagram.
What is the surface area of the new shape?



SOLUTION **38** cm^2

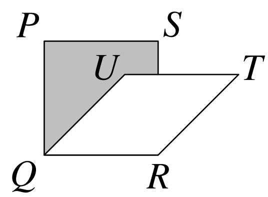
When two of the shapes are put together, the top and bottom faces both have area 7 cm^2. Also, there are four square faces of area 4 cm^2 and four rectangular faces of area 2 cm^2.
Hence the required surface area is $(2\times7+4\times4+4\times2)$ cm^2 $=(14+16+8)$ cm^2 $=38$ cm^2.

A8. Alex chooses three from the six primes 2003, 2011, 2017, 2027, 2029 and 2039.
The mean of his three primes is 2023.
What is the mean of the other three primes?

SOLUTION **2019**

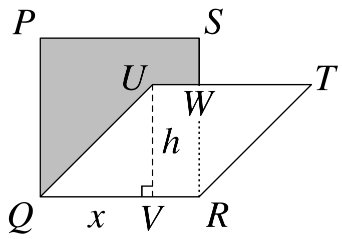
The sum of the six primes is $2003 + 2011 + 2017 + 2027 + 2029 + 2039 = 12\,126$.
The sum of the three primes which Alex chooses is $3 \times 2023 = 6069$.
Therefore the mean of the other three primes is $\dfrac{12\,126 - 6069}{3} = \dfrac{6057}{3} = 2019$.

A9. The diagram shows the square $PQRS$, which has area $25\,\text{cm}^2$, and the rhombus $QRTU$, which has area $20\,\text{cm}^2$.
What is the area of the shaded region?



SOLUTION $\mathbf{11}\,\text{cm}^2$

Let the perpendicular from U to QR meet QR at V and let the point where UT and SR intersect be W, as shown.
Let the lengths of UV and QV be h and x respectively.
Square $PQRS$ has area $25\,\text{cm}^2$, so its side-length is $5\,\text{cm}$.
Hence rhombus $QRTU$ has side-length $5\,\text{cm}$ and area $20\,\text{cm}^2$.



Therefore h is $\dfrac{20}{5}\,\text{cm} = 4\,\text{cm}$.

By Pythagoras' Theorem, $QU^2 = QV^2 + VU^2$. Therefore x is $\sqrt{5^2 - 4^2}\,\text{cm} = 3\,\text{cm}$.
So $UW = VR = (5 - 3)\,\text{cm} = 2\,\text{cm}$.
Hence the area of trapezium $UQRW$ is $\frac{1}{2} \times (2 + 5) \times 4\,\text{cm}^2 = 14\,\text{cm}^2$.
Therefore the area of the shaded region is $(25 - 14)\,\text{cm}^2 = 11\,\text{cm}^2$.

A10. What is the remainder when $\overbrace{23\cdots\cdots23}^{\text{twenty-three 23s}}$ is divided by 32?

Solution **3**

First note that $100\,000 = 10^5 = 2^5 \times 5^5$. So $100\,000$ is a multiple of 32.

Therefore the 46-digit number $\overbrace{23\cdots\cdots23}^{\text{twenty 23s}}200\,000$ is a multiple of 32.

Hence, when $\overbrace{23\cdots\cdots23}^{\text{twenty-three 23s}}$ is divided by 32, the remainder is equal to the remainder when $32\,323$ is divided by 32. As $32\,323 = 32 \times 1010 + 3$, the required remainder is 3.

Section B

B1. The sum of four fractions is less than 1. Three of these fractions are $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{10}$. The fourth fraction is $\frac{1}{n}$, where n is a positive integer. What values could n take?

Solution

We are given that $\frac{1}{2} + \frac{1}{3} + \frac{1}{10} + \frac{1}{n} < 1$. Therefore $\frac{15 + 10 + 3}{30} + \frac{1}{n} < 1$.

Hence $\frac{28}{30} + \frac{1}{n} < 1$. So $\frac{1}{n} < \frac{1}{15}$.

A unit fraction decreases in value when its numerator increases.
Hence $n > 15$. Therefore n could be any positive integer greater than 15.

B2. Laura went for a training ride on her bike. She covered the first 10% of the total distance in 20% of the total time of the ride. What was the ratio of her average speed over the first 10% of the distance to her average speed over the remaining 90% of the distance?

Solution

Let the distance Laura cycled be $10x$ and let the time it took her be $5t$. Then she cycled a distance x in a time t, followed by a distance $9x$ in a time $4t$.
So Laura's average speed over the first 10% of the distance was $\frac{x}{t}$ and her average speed over the remaining 90% of the distance was $\frac{9x}{4t}$.

Hence the ratio of the speeds is $\frac{x}{t} : \frac{9x}{4t} = 1 : \frac{9}{4} = 4 : 9$.

B3. As Rachel travelled to school, she noticed that, at each bus stop, one passenger got off and x passengers got on, where $x \geq 2$. After five stops, the number of passengers on the bus was x times the number of passengers before the first stop. How many passengers were on the bus before the first stop?

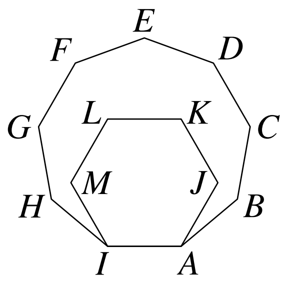
Solution

Let the number of passengers who were on the bus before the first stop be n.
Then, after five stops, the number of passengers on the bus was $n - 5 + 5x$.
So $n - 5 + 5x = nx$, that is, $nx - n = 5x - 5$. Therefore $n(x - 1) = 5(x - 1)$.

Hence $n = \frac{5(x-1)}{x-1} = 5$, as $x \neq 1$.

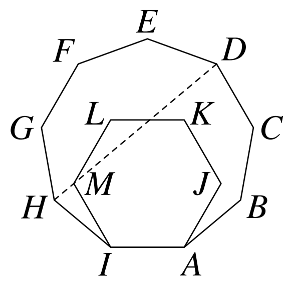
So there were five passengers on the bus before the first stop.

B4. The regular nonagon $ABCDEFGHI$ shares two of its vertices with the regular hexagon $AJKLMI$.
Show that the points H, M and D lie on the same straight line.



SOLUTION

First note that the exterior angles of a hexagon and nonagon are $\frac{360°}{6}$ and $\frac{360°}{9}$, that is, 60° and 40° respectively.
Hence the corresponding interior angles are 120° and 140° respectively.
Consider the hexagon $HIABCD$. The sum of its interior angles is $6 \times 120° = 720°$. Also $\angle HIA = \angle IAB = \angle ABC = \angle BCD = 140°$.
From the symmetry of the hexagon, we see that $\angle DHI = \angle CDH$.
So $\angle DHI = (720 - 4 \times 140)° \div 2 = 80°$.



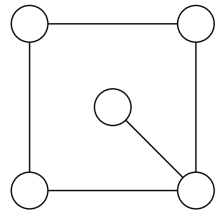
Now consider triangle HIM. Note that $\angle HIM = (140 - 120)° = 20°$.
The side-lengths of the regular nonagon and regular hexagon are equal, so $HI = MI$.
Hence triangle HIM is isosceles and $\angle MHI = \angle IMH = (180 - 20)° \div 2 = 80°$.
So $\angle MHI = \angle DHI$. Therefore H, M and D lie on the same straight line.

B5. The eleven-digit number '$A123456789B$' is divisible by exactly eight of the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9. Find the values of A and B, explaining why they must have these values.

SOLUTION

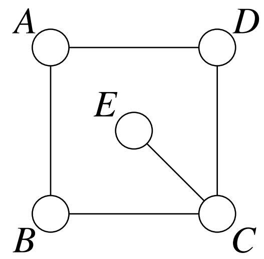
Let the eleven-digit number '$A123456789B$' be N.
First note that N is divisible by least two of 2, 4 and 8. Therefore it is divisible by 4. So B is not 5. A number is divisible by 4 if, and only if, its last two digits form a number divisible by 4. Hence B is not 0 as 90 is not divisible by 4. So N is not divisible by 5, but is divisible by each of the numbers 1, 2, 3, 4, 6, 7, 8, 9. A number is divisible by 8 if, and only if, its last three digits form a number divisible by 8. Therefore, for N to be divisible by 8, B is 6 as 896 is the only number between 890 and 899 inclusive which is divisible by 8. N is also divisible by 9, which means that the sum of its digits is also divisible by 9. Hence $A + 51$ is divisible by 9. Therefore A is 3 and, as has been shown, B is 6.
(*As* 31 234 567 896 *is divisible by both 8 and 9, it is clearly also divisible by 1, 2, 3, 4 and 6. It is left to the reader to check that* 31 234 567 896 *is also divisible by 7.*)

B6. The diagram shows five circles connected by five line segments. Three colours are available to colour these circles.
In how many different ways is it possible to colour all five circles so that circles which are connected by a line segment are coloured differently?



Solution

We first look at the number of different ways of colouring the four circles at the corners of the square.
If circles B and D are given the same colour, then they may be coloured in three different ways. For each of these, two colours are available for both circle A and circle C. Therefore, if B and D have the same colour, then the four corner squares may be coloured in $3 \times 2 \times 2 = 12$ different ways.



If circles B and D are coloured differently, then we can choose three colours for B and, for each of these, two different colours are available for D. So B and D may then be coloured in $2 \times 3 = 6$ different ways. Now, however, only one colour is available for circles A and C, so if circles B and D are coloured differently then there are six ways of colouring the corner circles.
Hence in total the four corner circles may be coloured in $12 + 6 = 18$ different ways.
For each of these 18 possibilities, circle E must be coloured differently from circle C and thus two colours are available for it. Therefore, in total, there are $2 \times 18 = 36$ different ways of colouring the five circles so that connected circles are coloured differently.